SyDe312 (Winter 2005)

Unit 1 - Solutions (continued)

March 1, 2005

Chapter 6 - Linear Systems

Problem 6.6 - 1b

Iterative solution by the Jacobi and Gauss-Seidel iteration methods:
Given: $b = [0 - 77]^T, \quad x = [0 - 11]^T$

$9x_1 + x_2 + x_3 = 0$
$2x_1 + 10x_2 + 3x_3 = -7$
$3x_1 + 4x_2 + 11x_3 = 7$

$x_1 = \frac{1}{9}(0 - x_2 - x_3)$
$x_2 = \frac{1}{10}(-7 - 2x_1 - 3x_3)$
$x_3 = \frac{1}{11}(7 - 3x_1 - 4x_2)$

Jacobi method:
$x_1^{(k+1)} = \frac{1}{9}(0 - x_2^{(k)} - x_3^{(k)})$
$x_2^{(k+1)} = \frac{1}{10}(-7 - 2x_1^{(k)} - 3x_3^{(k)})$
$x_3^{(k+1)} = \frac{1}{11}(7 - 3x_1^{(k)} - 4x_2^{(k)})$

Gauss-Seidel method:
$x_1^{(k+1)} = \frac{1}{9}(0 - x_2^{(k)} - x_3^{(k)})$
$x_2^{(k+1)} = \frac{1}{10}(-7 - 2x_1^{(k+1)} - 3x_3^{(k)})$
$x_3^{(k+1)} = \frac{1}{11}(7 - 3x_1^{(k+1)} - 4x_2^{(k+1)})$

For example beginiing with the initial $x^0 = [0,0,0]$, in the Jacobi method, we get:
$x_1^{(1)} = \frac{1}{9}(0-0-0) = 0$
$x_2^{(1)} = \frac{1}{10}(-7-2\times 0-3\times 0) = \frac{-7}{10}$
$x_3^{(1)} = \frac{1}{11}(7-3\times 0-4\times 0) = \frac{7}{11}$

and with the Gauss-Seidel method:
$x_1^{(1)} = \frac{1}{9}(0-0-0) = 0$
$x_2^{(1)} = \frac{1}{10}(-7-2\times 0-3\times 0) = \frac{-7}{10}$
$x_3^{(1)} = \frac{1}{11}(7-3\times 0-4\times \frac{-7}{10}) = \frac{98}{110}$

Iterating these procedure until $\left\|x - x^k\right\| \leq 0.00005$, we get:

k	x_1^k	x_2^k	x_3^k	Error	Ratio
1	0.0000E+0	-0.70000	0.63636	3.64E-1	0.364
2	7.0707E-3	-0.89091	0.89091	1.09E-1	0.300
3	-6.6227E-9	-0.96869	0.95840	4.16E-2	0.381
4	1.1427E-3	-0.98752	0.98861	1.25E-2	0.300
5	-1.2141E-4	-0.99681	0.99515	4.85E-3	0.389
6	1.8468E-4	-0.99852	0.99887	1.48E-3	0.305
7	-3.9246E-5	-0.99970	0.99941	5.88E-4	0.398
8	3.1941E-5	-0.99982	0.99990	1.84E-4	0.313
9	-9.5235E-6	-0.99998	0.99992	7.58E-5	0.411
10	5.8346E-6	-0.99998	0.99999	2.46E-5	0.325

Table 1: Problem 6.6.1b Jacobi Iteration

k	x_1^k	x_2^k	x_3^k	Error	Ratio
1	0.0000E+0	-0.7000	0.8909	3.00E-1	3.00E-1
2	-2.1212E-2	-0.9730	0.9923	3.70E-2	1.23E-1
3	-3.2568E-3	-0.9971	0.9998	3.26E-3	8.81E-2
4	-3.0720E-4	-0.9999	1.0000	3.07E-4	9.43E-2
5	-1.7557E-5	-1.0000	1.0000	1.76E-5	5.72E-2

Table 2: Problem 6.6.1b Gauss-Seidel Iteration

Problem 6.6 - 3

In case a, for the matrix $\begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$, in which the $\|M\| > 1$, the iteration methods do not converge.

k	x_1^k	x_2^k
1	1	0
2	1	-4
3	17	-4
4	17	-68
5	273	-68
6	273	-1092
7	4369	-1092
8	4369	-17476
9	69905	-17476
10	69905	-279620

Table 3: Problem 6.6.3a Jacobi Iteration

k	x_1^k	x_2^k
1	1	-4
2	17	-68
3	273	-1092
4	4369	-17476
5	69905	-279620
6	1118481	-4473924
7	17895697	-71582788
8	2.86E+08	-1.1453E9
9	4.58E+09	-1.8325E10
10	7.33E+10	-2.9320E11

Table 4: Problem 6.6.3a Gauss-Seidel Iteration

So the Gauss-Seidel method diverges more rapidly.

In case b, for the matrix $\begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, which is diagonally dominant, both methods converge.

So Gauss-Seidel method converges faster.

k	x_1^k	x_2^k	Error
1	0	0.25	0.0667
2	-0.0625	0.25	0.0167
3	-0.0625	0.2656	0.0042
4	-0.0664	0.2656	0.001
5	-0.0664	0.2666	2.60E-04
6	-0.0667	0.2666	6.51E-05
7	-0.0667	0.2667	1.63E-05
8	-0.0667	0.2667	4.07E-06

Table 5: Problem 6.6.3b Jacobi Iteration

k	x_1^k	x_2^k	Error
1	0.0000	0.25	6.67E-02
2	-0.0625	0.2656	4.20E-03
3	-0.0664	0.2666	2.60E-04
4	-0.0667	0.2667	6.51E-05
5	-0.0667	0.2667	1.02E-06

Table 6: Problem 6.6.3b Gauss-Seidel Iteration

Problem 6.6 - 4

$$A = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \quad b = \begin{bmatrix} 5 \\ -3 \\ -7 \\ 9 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

[The following solutions use the notation of the textbook for the iteration matrices etc. You can easily translate to relate these to the matrices defined in the lecture notes.]

Recall that in Jacobi:

$$N = \begin{bmatrix} a11 & 0 & \dots & 0 \\ 0 & a22 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & ann \end{bmatrix} \quad P = N - A$$

and In Gauss-Seidel:

$$N = \begin{bmatrix} a11 & 0 & \dots & 0 \\ a21 & a22 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ an1 & an2 & \dots & ann \end{bmatrix} \quad P = N - A$$

and $M = N^{-1}P$

k	x_1^k	x_2^k	x_3^k	x_4^k	Error	Ratio
1	1.25	-0.75	-1.75	2.25	7.50E-01	0.375
2	0.625	0.125	-0.875	1.625	3.75E-01	0.5
3	1.0625	-0.1875	-1.1875	2.0625	1.88E-01	0.5
4	0.9063	0.0313	-0.9688	1.9063	9.38E-02	0.5
5	1.0156	-0.0469	-1.0469	2.0156	4.69E-02	0.5
6	0.9766	0.0078	-0.9922	1.9766	2.34E-02	0.5
7	1.0039	-0.0117	-1.0117	2.0039	1.17E-02	0.5
8	0.9941	0.002	-0.998	1.9941	5.86E-03	0.5
9	1.001	-0.0029	-1.0029	2.001	2.93E-03	0.5
10	0.9985	0.0005	-0.9995	1.9985	1.46E-03	0.5
11	1.0002	-0.0007	-1.0007	2.0002	7.32E-04	0.5
12	0.9996	0.0001	-0.9999	1.9996	3.66E-04	0.5
13	1.0001	-0.0002	-1.0002	2.0001	1.83E-04	0.5
14	0.9999	0.0000	-1.0000	1.9999	9.16E-05	0.5
15	1.0000	0.0000	-1.0000	2.0000	4.58E-05	0.5

Table 7: Problem 6.6 - 4 Jacobi Iteration

$$M = \frac{1}{4}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \qquad \|M\|_\infty = 0.5.$$

This is consistent with the column ratio in the above table.

Problem 6.6 - 4b

$$M = \frac{1}{4}\begin{bmatrix} 0 & 1/4 & 1/4 & 0 \\ 0 & 1/16 & 1/16 & 1/4 \\ 0 & 1/16 & 1/16 & 1/4 \\ 0 & 1/32 & 1/32 & 1/8 \end{bmatrix} \qquad \|M\|_\infty = 0.5.$$

k	x_1^k	x_2^k	x_3^k	x_4^k	Error	Ratio
1	1.2500	-4.38E-01	-1.44E+00	1.7813	4.38E-01	0.219
2	0.7813	-1.09E-01	-1.11E+00	1.9453	2.19E-01	0.500
3	0.9453	-2.73E-02	-1.03E+00	1.9863	5.47E-02	0.250
4	0.9863	-6.84E-03	-1.01E+00	1.9966	1.37E-02	0.250
5	0.9966	-1.71E-03	-1.00E+00	1.9991	3.42E-03	0.250
6	0.9991	-4.27E-04	-1.00E+00	1.9998	8.54E-04	0.250
7	0.9998	-1.07E-04	-1.00E+00	1.9999	2.14E-04	0.250
8	0.9999	-2.67E-05	-1.00E+00	2.0000	5.34E-05	0.250
9	1.0000	-6.68E-06	-1.00E+00	2.0000	1.34E-05	0.250

Table 8: Problem 6.6 - 4 Gauss-Seidel Iteration

The actual convergence rate is better than 0.5. Keep in mind that $||M||$ gives an upper bound on the convergence or divergence rate.

Problem 6.6 - 5

$$\begin{bmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

6.6 - 5a

$$\begin{vmatrix} 4 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{vmatrix} = 23$$

since $\det(A) \neq 0$, for any right-hand side vector b, the system has a unique solution $x = [x_1, x_2, x_3]^T$.

6.6 - 5b

(1)Jacobi method:

$$N = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad N^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} \quad P = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

$$M = N^{-1}P = \begin{bmatrix} 0 & -1/4 & -1/4 \\ -1/2 & 0 & 0 \\ 0 & -2/3 & 0 \end{bmatrix}$$

$$\|M\|_\infty = 2/3$$

Since $\|M\|_\infty < 1$, the method converges.

(2) Gauss-Seidel method:

$$N = \begin{bmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 2 & 3 \end{bmatrix} \quad N^{-1} = \begin{bmatrix} 1/4 & 0 & 0 \\ -1/8 & 1/2 & 0 \\ 1/12 & -1/3 & 1/3 \end{bmatrix} \quad P = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M = N^{-1}P = \begin{bmatrix} 0 & -1/4 & -1/4 \\ 0 & 1/8 & 1/8 \\ 0 & -1/12 & -1/12 \end{bmatrix}$$

$$\|M\|_\infty = 1/2$$

Since $\|M\|_\infty < 1$, the method converges.

Problem 6.6 - 6

Consider $A_n = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & ... & 0 & -1 & 2 \end{bmatrix}$

The iteration method $Nx^{(k+1)} = b + Px^{(k)}$ will converge, for all b vectors and all initial guesses $x^{(0)}$, if and only if all eigenvalues λ of $M = N^{-1}P$ satisfy

$$|\lambda| < 1$$

(1) Jacobi method

n=5,

```
A=[2,-1,0,0,0;-1,2,-1,0,0;0,-1,2,-1,0;0,0,-1,2,-1;0,0,0,-1,2];
for i=1:n
 for j=1:n
 N_Jacobi(i,j)=(i==j)*A(i,j);
 end
end
P=N_Jacobi-A;
M=inv(N_Jacobi)*P;
lambda=eig(M)
```

$$A_5 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$N = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \quad N^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$M = N^{-1}P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

The eigenvalues of M are - using eig(M):

$$\lambda_1 = -0.866 \quad \lambda_2 = -0.5 \quad \lambda_3 = 0$$
$$\lambda_4 = 0.5 \quad \lambda_5 = 0.866$$

Likewise for $n = 10$, the eigenvalues of M are:

$\lambda_1 = -0.9595 \quad \lambda_2 = -0.8413 \quad \lambda_3 = -0.6549 \quad \lambda_4 = -0.4154 \quad \lambda_5 = -0.1423$
$\lambda_6 = 0.1423 \quad \lambda_7 = 0.4154 \quad \lambda_8 = 0.6549 \quad \lambda_9 = 0.8413 \quad \lambda_{10} = 0.9595$

For $n = 20$, the eigenvalues of M are:

$\lambda_1 = -0.9888 \quad \lambda_2 = -0.9556 \quad \lambda_3 = -0.9010 \quad \lambda_4 = -0.8262 \quad \lambda_5 = -0.7331$
$\lambda_6 = -0.6235 \quad \lambda_7 = -0.5000 \quad \lambda_8 = -0.3653 \quad \lambda_9 = -0.2225 \quad \lambda_{10} = -0.0747$
$\lambda_{11} = 0.0747 \quad \lambda_{12} = 0.2225 \quad \lambda_{13} = 0.3653 \quad \lambda_{14} = 0.5000 \quad \lambda_{15} = 0.6235$
$\lambda_{16} = 0.7331 \quad \lambda_{17} = 0.8262 \quad \lambda_{18} = 0.9010 \quad \lambda_{19} = 0.9556 \quad \lambda_{20} = 0.9888$

We see that for $n = 5, 10, 20$, $|\lambda_i| < 1$, so Jacobi method will converge.

(2) Gauss-Seidel method

Again for $n = 5$, we have:

```
for i=1:n
 for j=1:n
 N_GS(i,j)=(i>=j)*A(i,j);
 end
end
P=N_GS-A;
M=inv(Ngs)*P;
lambda=eig(M)
```

$$M = N^{-1}P = \begin{bmatrix} 0 & 0.5000 & 0 & 0 & 0 \\ 0 & 0.2500 & 0.5000 & 0 & 0 \\ 0 & 0.1250 & 0.2500 & 0.5000 & 0 \\ 0 & 0.0625 & 0.1250 & 0.2500 & 0.5000 \\ 0 & 0.0313 & 0.0625 & 0.1250 & 0.2500 \end{bmatrix}$$

The eigenvalues of M are - using eig(M):

$$\lambda_1 = 0 \quad \lambda_2 = 0.75 \quad \lambda_3 = 0.25$$
$$\lambda_4 = 0 \quad \lambda_5 = 0$$

Likewise for $n = 10$, the eigenvalues of M are:

$$\lambda_1 = 0 \quad \lambda_2 = -0.9206 \quad \lambda_3 = -0.7077 \quad \lambda_4 = -0.4288 \quad \lambda_5 = -0.1726$$
$$\lambda_6 = 0.0203 \quad \lambda_7 = 0 \quad \lambda_8 = 0 \quad \lambda_9 = 0 \quad \lambda_{10} = 0$$

For $n = 20$, the eigenvalues of M are:

$\lambda_1 = 0 \quad \lambda_2 = -0.9778 \quad \lambda_3 = -0.9131 \quad \lambda_4 = -0.8117 \quad \lambda_5 = -0.6827$
$\lambda_6 = -0.5374 \quad \lambda_7 = -0.3887 \quad \lambda_8 = -0.2500 \quad \lambda_9 = -0.1335 \quad \lambda_{10} = -0.0495$
$\lambda_{11} = 0.0068 \quad \lambda_{12} = 0.0051 + 0.0046i \quad \lambda_{13} = 0.0051 - 0.0046i$
$\lambda_{14} = 0.0008 + 0.0071i \quad \lambda_{15} = 0.0008 - 0.0071i \quad \lambda_{16} = -0.0043 + 0.0061i$
$\lambda_{17} = -0.0043 - 0.0061i \quad \lambda_{18} = -0.0078 + 0.0023i \quad \lambda_{19} = -0.0078 + 0.0023i \quad \lambda_{20} = 0$

So for $n = 5, 10, 20$, $|\lambda_i| < 1$, so Gauss-Seidel method will converge.

Problem 6.6 - 14

Note: this is slightly different from the method described in the lecture notes, but equivalent. Careful with the sign of the correction vector.

The residual correction method:

$A_\epsilon x = b$ where $A_\epsilon = A_0 + \epsilon B$.

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Let $N = A_0$ and $P = N - A_\epsilon = -\epsilon B$, so $M = N^{-1}P = A_0^{-1}(-\epsilon B)$

$$\|M\| = \left\|A_0^{-1}(-\epsilon B)\right\| = |\epsilon|.\left\|A_0^{-1}B)\right\|$$

$$= |\epsilon| \left\| \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ -0.50 & 0.00 & 0.50 \\ 0.25 & -0/50 & -0.25 \end{bmatrix} \right\| = |\epsilon|.1 = |\epsilon|$$

In order to ensure the convergence of the residual correction method, $\alpha < 1$.

k	x_1^k	x_2^k	x_3^k	Error	Ratio
1	1.1000e+000	1.0000e+000	9.0000e-001	1.0000e-001	9.0909e-002
2	1.0000e+000	1.0100e+000	1.0000e+000	1.0000e-002	1.0000e-001
3	9.9950e-001	1.0000e+000	1.0005e+000	5.0000e-004	5.0000e-002
4	1.0000e+000	9.9995e-001	1.0000e+000	5.0000e-005	1.0000e-001
5	1.0000e+000	1.0000e+000	1.0000e+000	2.5000e-006	5.0000e-002

Table 9: Problem 6.6.14 $\epsilon = 0.1$

In this case, we are not given the vector b, but the true answer $x = [1, 1, 1]$. Let $x^0 = [0, 0, 0]$ and repeat the procedure until $\left\|x - x^{(k)}\right\| \leq 10^{-5}$

k	x_1^k	x_2^k	x_3^k	Error	Ratio
1	1.2000e+000	1.0000e+000	8.0000e-001	2.0000e-001	1.6667e-001
2	1.0000e+000	1.0400e+000	1.0000e+000	4.0000e-002	2.0000e-001
3	9.9600e-001	1.0000e+000	1.0040e+000	4.0000e-003	1.0000e-001
4	1.0000e+000	9.9920e-001	1.0000e+000	8.0000e-004	2.0000e-001
5	1.0001e+000	1.0000e+000	9.9992e-001	8.0000e-005	1.0000e-001
6	1.0000e+000	1.0000e+000	1.0000e+000	1.6000e-005	2.0000e-001
7	1.0000e+000	1.0000e+000	1.0000e+000	1.6000e-006	1.0000e-001

Table 10: Problem 6.6.14 $\epsilon = 0.2$

The number of iterations increases as ϵ increases, consistent with expectations.

Problem 7.2 - 1b

For $\begin{bmatrix} 2 & 36 \\ 36 & 23 \end{bmatrix}$,

$f(\lambda) = \lambda^2 - 25\lambda - 1250 = (\lambda - 50)(\lambda + 251)$,

$\lambda_1 = 50, \lambda_2 = -25$

Using $(A - \lambda I)v = 0$, we get :

An eigenvector corresponding to $\lambda = 50$: $v_1 = [2, 5]^T$

k	x_1^k	x_2^k	x_3^k	Error	Ratio
1	1.5000e+000	1.0000e+000	5.0000e-001	5.0000e-001	3.3333e-001
2	1.0000e+000	1.2500e+000	1.0000e+000	2.5000e-001	5.0000e-001
3	9.3750e-001	1.0000e+000	1.0625e+000	6.2500e-002	2.5000e-001
4	1.0000e+000	9.6875e-001	1.0000e+000	3.1250e-002	5.0000e-001
5	1.0078e+000	1.0000e+000	9.9219e-001	7.8125e-003	2.5000e-001
6	1.0000e+000	1.0039e+000	1.0000e+000	3.9063e-003	5.0000e-001
7	9.9902e-001	1.0000e+000	1.0010e+000	9.7656e-004	2.5000e-001
8	1.0000e+000	9.9951e-001	1.0000e+000	4.8828e-004	5.0000e-001
9	1.0001e+000	1.0000e+000	9.9988e-001	1.2207e-004	2.5000e-001
10	1.0000e+000	1.0001e+000	1.0000e+000	6.1035e-005	5.0000e-001
11	9.9998e-001	1.0000e+000	1.0000e+000	1.5259e-005	2.5000e-001
12	1.0000e+000	9.9999e-001	1.0000e+000	7.6294e-006	5.0000e-001

Table 11: Problem 6.6.14 $\epsilon = 0.5$

k	x_1^k	x_2^k	x_3^k	Error	Ratio
1	1.80E+00	1.00E+00	2.00E-01	8.00E-01	4.44E-01
2	1.00E+00	1.64E+00	1.00E+00	6.40E-01	8.00E-01
3	7.44E-01	1.00E+00	1.26E+00	2.56E-01	4.00E-01
4	1.00E+00	7.95E-01	1.00E+00	2.05E-01	8.00E-01
5	1.08E+00	1.00E+00	9.18E-01	8.19E-02	4.00E-01
6	1.00E+00	1.07E+00	1.00E+00	6.55E-02	8.00E-01
7	9.74E-01	1.00E+00	1.03E+00	2.62E-02	4.00E-01
8	1.00E+00	9.79E-01	1.00E+00	2.10E-02	8.00E-01
9	1.01E+00	1.00E+00	9.92E-01	8.39E-03	4.00E-01
10	1.00E+00	1.01E+00	1.00E+00	6.71E-03	8.00E-01
11	9.97E-01	1.00E+00	1.00E+00	2.68E-03	4.00E-01
12	1.00E+00	9.98E-01	1.00E+00	2.15E-03	8.00E-01
13	1.00E+00	1.00E+00	9.99E-01	8.59E-04	4.00E-01
14	1.00E+00	1.00E+00	1.00E+00	6.87E-04	8.00E-01
15	1.00E+00	1.00E+00	1.00E+00	2.75E-04	4.00E-01
16	1.00E+00	1.00E+00	1.00E+00	2.20E-04	8.00E-01
17	1.00E+00	1.00E+00	1.00E+00	8.80E-05	4.00E-01
18	1.00E+00	1.00E+00	1.00E+00	7.04E-05	8.00E-01
19	1.00E+00	1.00E+00	1.00E+00	2.81E-05	4.00E-01
20	1.00E+00	1.00E+00	1.00E+00	2.25E-05	8.00E-01
21	1.00E+00	1.00E+00	1.00E+00	9.01E-06	4.00E-01

Table 12: Problem 6.6.14 $\epsilon = 0.8$

An eigenvector corresponding to $\lambda = -25$: $v_2 = [1, -1]^T$

Problem 7.2 - 12

For the symmetric matrix $A = \begin{bmatrix} -7 & 13 & -16 \\ 13 & -10 & 23 \\ -16 & 13 & -7 \end{bmatrix}$, eigenvalues and eigenvectors are:

$\lambda = \{-36, 3, 9\}$

$v^{(1)} = [\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}]^T, \quad v^{(2)} = [\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}]^T, \quad v^{(3)} = [\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{2}}]^T, \quad ,$

U can be: $U = [v^{(1)}; v^{(2)}; v^{(3)}] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Each column in U is one of the eigenvectors of A. The eigenvectors are normalized to get the columns of U which is supposed to be an orthogonal matrix.

We get: $U^T AU = D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ and $U^T U = UU^T = I$.

```
A=[-7,13,-16; 13, -10, 13; -16, 13, -7];
[U,lam]=eig(A)
```

The result might be:

$$\lambda = \begin{bmatrix} -36.0000 & 0 & 0 \\ 0 & 3.0000 & 0 \\ 0 & 0 & 9.0000 \end{bmatrix}, \quad U = \begin{bmatrix} -0.5774 & 0.4082 & 0.7071 \\ 0.5774 & 0.8165 & -0.0000 \\ -0.5774 & 0.4082 & -0.7071 \end{bmatrix} = [v^{(1)}; v^{(3)}; v^{(2)}]$$

Run the following command, in order to see:

```
U'*A*U
```

$$\begin{bmatrix} -36.0000 & 0 & 0 \\ 0 & 3.0000 & 0 \\ 0 & 0 & 9.0000 \end{bmatrix}$$

and also the commands `U'*U` *and* `U*U'`, in order to see $U^T U = UU^T = I$.

Problem 7.2 - 14

For $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, eigenvalues and eigenvectors are:

$f(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \Rightarrow$:

$\lambda_1 = 1, v^{(1)} = [1, 0]^T$

$\lambda_2 = 1, v^{(1)} = [0, 1]^T$

For $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, e-values and e-vectors are:

$f(\lambda) = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 \Rightarrow$:

$\lambda_1 = 1, v^{(1)} = [0, 1]^T$

In case b, this eigenvector and its multiples are the only eigenvectors of the matrix. It does NOT contradict Theorem 7.2.4, because the matrix is not symmetric.